1984 AIME

Time limit: 180 minutes

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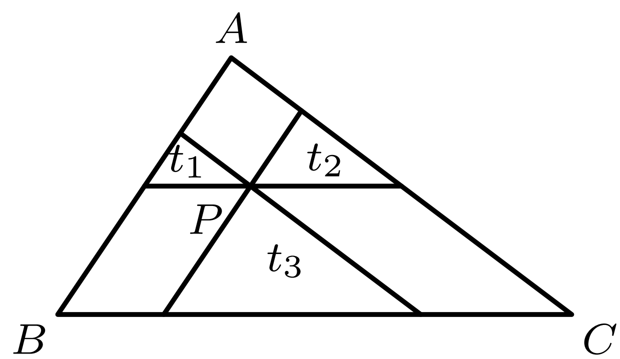
1. Find the value of $a_2 + a_4 + a_6 + \cdots + a_{98}$ if $a_1, a_2, a_3, \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \cdots + a_{98} = 137$.

2. The integer n is the smallest positive multiple of 15 such that every digit of n is either 0 or 8. Compute $\frac{n}{15}$.

3. A point P is chosen in the interior of $\triangle ABC$ so that when lines are drawn through P parallel to the sides of $\triangle ABC$, the resulting smaller triangles, t_1, t_2, and t_3 in the figure, have areas $4, 9$, and 49, respectively. Find the area of $\triangle ABC$.



4. Let S be a list of positive integers—not necessarily distinct—in which the number 68 appears. The average (arithmetic mean) of the numbers in S is 56. However, if 68 is removed, the average of the remaining numbers drops to 55. What is the largest number that can appear in S?

5. Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.

6. Three circles, each of radius 3, are drawn with centers at $(14, 92)$, $(17, 76)$, and $(19, 84)$. A line passing through $(17, 76)$ is such that the total area of the parts of the three circles to one side of the line is equal to the total area of the parts of the three circles to the other side of it. What is the absolute value of the slope of this line?

7. The function f is defined on the set of integers and satisfies $f(n) = n - 3$ if $n \geq 1000$, and

$$f(n) = f(f(n+5))$$

if $n < 1000$. Find $f(84)$.

8. The equation $z^6 + z^3 + 1 = 0$ has one complex root with argument θ between 90° and 180° in the complex plane. Determine the degree measure of θ.

9. In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is $15\ \mathrm{cm}^2$ and the area of face ABD is $12\ \mathrm{cm}^2$. These two faces meet each other at a 30° angle. Find the volume of the tetrahedron in cm^3.

10. Mary told John her score on the American High School Mathematics Examination (AHSME), which was over 80. From this, John was able to determine the number of problems Mary solved correctly. If Mary's score had been any lower, but still over 80, John could not have determined this. What was Mary's score? (Recall that the AHSME consists of 30 multiple-choice problems and that one's score, s, is computed by the formula $s = 30 + 4c - w$, where c is the number correct and w is the number wrong; students are not penalized for problems left unanswered.)

11. A gardener plants three maple trees, four oak trees, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let $\frac{m}{n}$ in lowest terms be the probability that no two birch trees are next to one another. Find $m + n$.

12. A function f is defined for all real numbers and satisfies

$$f(2+x) = f(2-x),$$
$$f(7+x) = f(7-x)$$

for all real x. If $x = 0$ is a root of $f(x) = 0$, what is the least number of roots $f(x) = 0$ must have in the interval $-1000 \leq x \leq 1000$?

13. Find the value of

$$10\cot\big(\cot^{-1}3+\cot^{-1}7 \\ +\cot^{-1}13+\cot^{-1}21\big).$$

14. What is the largest even integer that cannot be written as the sum of two odd composite numbers? (Recall that a positive integer is said to be composite if it is divisible by at least one positive integer other than 1 and itself.)

15. Determine $w^2+x^2+y^2+z^2$ if

$$\frac{x^2}{2^2-1}+\frac{y^2}{2^2-3^2}+\frac{z^2}{2^2-5^2}+\frac{w^2}{2^2-7^2}=1,$$

$$\frac{x^2}{4^2-1}+\frac{y^2}{4^2-3^2}+\frac{z^2}{4^2-5^2}+\frac{w^2}{4^2-7^2}=1,$$

$$\frac{x^2}{6^2-1}+\frac{y^2}{6^2-3^2}+\frac{z^2}{6^2-5^2}+\frac{w^2}{6^2-7^2}=1,$$

$$\frac{x^2}{8^2-1}+\frac{y^2}{8^2-3^2}+\frac{z^2}{8^2-5^2}+\frac{w^2}{8^2-7^2}=1.$$

Solutions: https://live.poshenloh.com/past-contests/aime/1984/solutions