1983 AIME Solutions

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1. Let $x, y,$ and z all exceed 1 and let w be a positive number such that

$$\begin{aligned} \log_x w &= 24, \\ \log_y w &= 40, \\ \log_{xyz} w &= 12. \end{aligned}$$

Find $\log_z w$.

Solution:

Taking reciprocals of the given logarithms gives

$$\begin{aligned} \log_w x &= \frac{1}{24}, \\ \log_w y &= \frac{1}{40}, \\ \log_w(xyz) &= \frac{1}{12}. \end{aligned}$$

Therefore

$$\log_w z = \frac{1}{12} - \frac{1}{24} - \frac{1}{40} = \frac{1}{60}.$$

Taking the reciprocal yields $\log_z w = 60$.

2. Let $f(x)=|x-p|+|x-15|+|x-p-15|$, where $0<p<15$. Determine the minimum value taken by $f(x)$ for x in the interval $p\leq x\leq 15$.

Solution:

Since $p\leq x\leq 15$, we have

$$|x-p|=x-p,$$
$$|x-15|=15-x,$$
$$|x-p-15|=p+15-x.$$

Hence $f(x)=30-x$. This is minimized at the right endpoint $x=15$, where its value is 15.

3. What is the product of the real roots of the equation

$$x^2+18x+30$$
$$=2\sqrt{x^2+18x+45}\,?$$

Solution:

Set $t=\sqrt{x^2+18x+45}$, so $t\geq 0$. The equation becomes

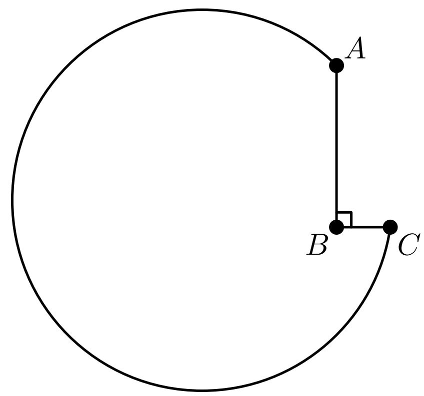
$$t^2-15=2t,$$

or $(t-5)(t+3)=0$. Thus $t=5$, and

$$x^2+18x+45=25,$$

so the real roots satisfy $x^2+18x+20=0$. By Vieta's formulas, their product is 20.

4. A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm, and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.



Solution:

Put $B = (0,0)$, $A = (0,6)$, and $C = (2,0)$. The midpoint of AC is $M = (1,3)$, and $AC = \sqrt{40}$. If O is the center, then

$$\begin{aligned} OM &= \sqrt{OA^2 - AM^2} \\ &= \sqrt{50 - 10} = \sqrt{40}. \end{aligned}$$

A vector perpendicular to $AC = (2,-6)$ is $(6,2)$, which already has length $\sqrt{40}$. Thus the two possible centers are

$$\begin{aligned} M + (6,2) &= (7,5), \\ M - (6,2) &= (-5,1). \end{aligned}$$

The pictured notched circle has its center on the side opposite the notch, so $O = (-5,1)$. Therefore $BO^2 = (-5)^2 + 1^2 = 26.$

5. Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x + y$ can have?

Solution:

Let $s = x + y$ and $p = xy$. From $x^2 + y^2 = s^2 - 2p = 7$, we get $p = \frac{s^2-7}{2}$. Also,

$$\begin{aligned} 10 &= x^3 + y^3 \\ &= s^3 - 3ps \\ &= s^3 - \frac{3s(s^2-7)}{2}. \end{aligned}$$

Hence

$$\begin{aligned} s^3 - 21s + 20 &= 0, \\ (s-1)(s-4)(s+5) &= 0. \end{aligned}$$

Each root gives a possible pair of complex roots of $t^2 - st + p = 0$, so the largest real possible value of $s = x + y$ is 4.

6. Let $a_n = 6^n + 8^n$. Determine the remainder on dividing a_{83} by 49.

Solution:

Modulo 49, every binomial term containing 7^2 vanishes. Since 83 is odd,

$$6^{83} = (7-1)^{83} \equiv -1 + 83 \cdot 7$$

and

$$8^{83} = (7+1)^{83} \equiv 1 + 83 \cdot 7.$$

Their sum is congruent to $166 \cdot 7 = 1162 \equiv 35 \pmod{49}$.

7. Twenty five of King Arthur's knights are seated at their customary round table. Three of them are chosen - all choices of three being equally likely - and are sent off to slay a troublesome dragon. Let P be the probability that at least two of the three had been sitting next to each other. If P is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

Solution:

There are $\binom{25}{3} = 2300$ selections. Fix one seat. If it is not selected, choosing three nonadjacent seats among the remaining 24 seats is equivalent to choosing three nonconsecutive positions from a row of 24, giving $\binom{22}{3}$. If the fixed seat is selected, its two neighbors are forbidden, and the other two selected seats must be nonconsecutive among the remaining row of 22, giving $\binom{21}{2}$.

Thus the number with no adjacent selected seats is

$$\begin{aligned}\binom{22}{3} + \binom{21}{2} &= 1540 + 210 \\ &= 1750.\end{aligned}$$

Therefore

$$P = 1 - \frac{1750}{2300} = \frac{11}{46},$$

and the requested sum is $11 + 46 = 57$.

8. What is the largest 2-digit prime factor of the integer $n = \binom{200}{100}$?

Solution:

For a prime $p > 50$, we have $p^2 > 200$, so

$$v_p\left(\binom{200}{100}\right) = \left\lfloor \frac{200}{p} \right\rfloor - 2\left\lfloor \frac{100}{p} \right\rfloor.$$

If $67 \leq p < 100$, this is $2 - 2 = 0$. If $50 < p \leq 66$, it is $3 - 2 = 1$. Hence no prime greater than 66 divides the binomial coefficient, while every prime between 50 and 66 does. The largest such prime is 61.

9. Find the minimum value of

$$\frac{9x^2 \sin^2 x + 4}{x \sin x}$$

for $0 < x < \pi$.

Solution:

Let $t = x \sin x > 0$. The expression becomes

$$9t + \frac{4}{t} \geq 2\sqrt{9t \cdot \frac{4}{t}} = 12,$$

with equality when $9t = \frac{4}{t}$, or $t = \frac{2}{3}$. The continuous function $x \sin x$ tends to 0 as x tends to 0 and equals $\frac{\pi}{2} > \frac{2}{3}$ at $x = \frac{\pi}{2}$. Thus it takes the value $\frac{2}{3}$ somewhere in $(0, \frac{\pi}{2})$, so the lower bound 12 is attained.

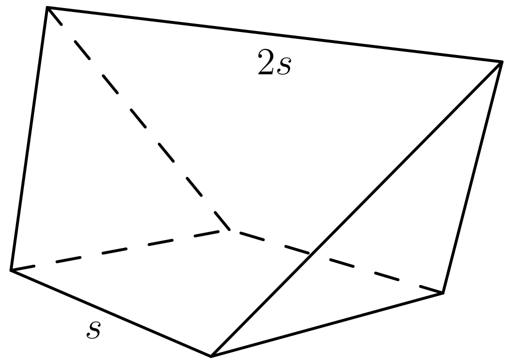
10. The numbers $1447, 1005,$ and 1231 have something in common: each is a four-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

Solution:

If 1 is the repeated digit, exactly one of the last three positions contains $1.$ There are 3 choices for that position, 9 choices for the next digit, and 8 for the last digit, since those two digits must differ from each other and from $1.$ This gives $3 \cdot 9 \cdot 8 = 216$ numbers.

If a digit other than 1 is repeated, there are 9 choices for that digit, 3 ways to choose its two positions among the last three, and 8 choices for the remaining digit. This gives another $9 \cdot 3 \cdot 8 = 216$ numbers. The total is $216 + 216 = 432.$

11. The solid shown has a square base of side length s. The upper edge is parallel to the base and has length $2s$. All other edges have length s. Given that $s = 6\sqrt{2}$, what is the volume of the solid?



Solution:

Each endpoint of the upper edge is joined to the endpoints of one side of the square base. Its distance to that side's midpoint is therefore $\frac{\sqrt{3}s}{2}$. The projection of each upper endpoint lies $\frac{s}{2}$ beyond that midpoint, because the two projected endpoints are $2s$ apart while the two opposite-side midpoints are s apart. Thus the height h satisfies

$$h^2 = \left(\frac{\sqrt{3}s}{2}\right)^2 - \left(\frac{s}{2}\right)^2 = \frac{s^2}{2},$$

so $h = \frac{s}{\sqrt{2}}$.

At a fraction t of the height above the base, the cross-section is a rectangle whose dimensions are $s(1-t)$ and $s(1+t)$. Its area is $s^2(1-t^2)$. By Cavalieri's principle, the volume is the volume s^2h of a prism minus the volume $\frac{s^2h}{3}$ of a pyramid:

$$V = \frac{2}{3}s^2h = \frac{\sqrt{2}}{3}s^3.$$

Substituting $s = 6\sqrt{2}$ gives $V = 288$.

12. Diameter AB of a circle has length a 2-digit integer (base ten). Reversing the digits gives the length of the perpendicular chord CD. The distance from their intersection point H to the center O is a positive rational number. Determine the length of AB.

Solution:

Let the diameter be $10a + b$ and the chord be $10b + a$. Since a perpendicular from the center bisects a chord,

$$\begin{aligned} 4OH^2 &= (10a+b)^2 \\ &\quad - (10b+a)^2 \\ &= 99(a^2 - b^2). \end{aligned}$$

For OH to be rational, $11(a^2 - b^2)$ must be a square. Thus it equals $121c^2$, so

$$(a-b)(a+b) = 11c^2.$$

Because a and b are digits with $a > b$, the left side is at most 81, so $c = 1$ or 2. If $c = 1$, the factors are 1 and 11, giving $a = 6$ and $b = 5$. If $c = 2$, the factor pairs of 44 either have different parity or give $a > 9$. Hence the diameter is 65.

13. For $\{1, 2, 3, \ldots, n\}$ and each of its nonempty subsets a unique **alternating sum** is defined as follows: Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (For example, the alternating sum for $\{1, 2, 4, 6, 9\}$ is $9 - 6 + 4 - 2 + 1 = 6$ and for $\{5\}$ it is simply 5.) Find the sum of all such alternating sums for $n = 7$.

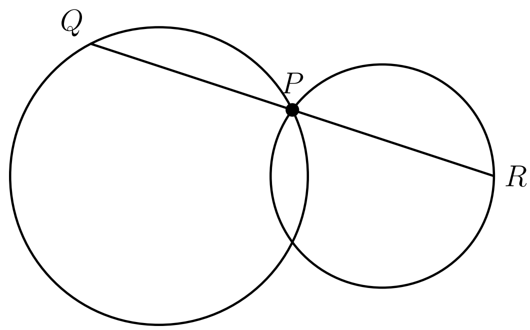
Solution:

Fix k. Once k is included, the $k - 1$ smaller elements may be chosen arbitrarily, contributing a factor of 2^{k-1}. If exactly j of the $7 - k$ larger elements are chosen, the sign of k is $(-1)^j$. Therefore the coefficient of k in the total is

$$\begin{aligned} 2^{k-1} \sum_{j=0}^{7-k} (-1)^j \binom{7-k}{j} \\ = 2^{k-1}(1-1)^{7-k}. \end{aligned}$$

This is 0 for $k < 7$ and 2^6 for $k = 7$. Hence the total is $7 \cdot 2^6 = 448$.

14. In the adjoining figure, two circles of radii 6 and 8 are drawn with their centers 12 units apart. At P, one of the points of intersection, a line is drawn in such a way that the chords QP and PR have equal length. Find the square of the length of QP.



Solution:

Put the center of the radius-8 circle at $O_1 = (-6, 0)$ and the other center at $O_2 = (6, 0)$. Subtracting the two circle equations gives

$$P = \left(\frac{7}{6}, \frac{\sqrt{455}}{6}\right),$$

where the positive y-coordinate selects the pictured intersection.

Let the common chord length be ℓ, and let u be the unit vector from Q toward R. Then $Q = P - \ell u$ and $R = P + \ell u$. Comparing Q with P in the first circle and R with P in the second gives

$$u \cdot (P - O_1) = \frac{\ell}{2},$$
$$u \cdot (P - O_2) = -\frac{\ell}{2}.$$

Adding shows that $u \perp P$, so

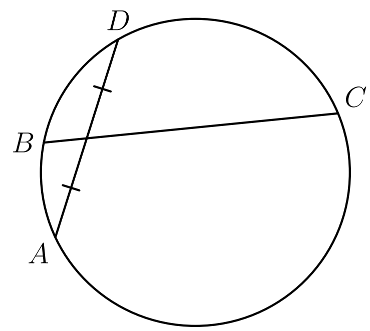
$$u = \frac{(\sqrt{455}, -7)}{\sqrt{504}}.$$

Subtracting the two dot-product equations gives

$$\ell = u \cdot (O_2 - O_1) = \frac{12\sqrt{455}}{\sqrt{504}}.$$

Therefore $\ell^2 = \frac{144 \cdot 455}{504} = 130.$

15. The adjoining figure shows two intersecting chords in a circle, with B on minor arc AD. Suppose that the radius of the circle is 5, that $BC = 6$, and that AD is bisected by BC. Suppose further that AD is the only chord starting at A which is bisected by BC. It follows that the sine of the minor arc AB is a rational number. If this fraction is expressed as a fraction $\frac{m}{n}$ in lowest terms, what is the product mn?



Solution:

Let H be the midpoint of AD. Put $H = (0,0), A = (-a,0), D = (a,0)$, and let the circle's center be $O = (0,k)$. Then $a^2 + k^2 = 25$. The midpoints of all chords starting at A form the circle with center $\frac{A+O}{2}$ and radius $\frac{5}{2}$. Because AD is the only such chord bisected by the line BC, that line is tangent to the midpoint circle at H.

Hence a unit direction vector for BC may be taken as $u = \frac{(k,a)}{5}$, perpendicular to $A + O = (-a,k)$. Write $B = -pu$ and $C = qu$, where $p, q > 0$. Then $p + q = 6$, and intersecting chords give $pq = a^2$. Substituting the line into the original circle also gives

$$q - p = 2u \cdot O = \frac{2ak}{5}.$$

Therefore

$$\begin{aligned} 36 - 4a^2 &= (q-p)^2 \\ &= \frac{4a^2(25-a^2)}{25}, \end{aligned}$$

so $a^4 - 50a^2 + 225 = 0$. Its roots are $a^2 = 5$ and 45, but $a^2 = pq \leq \frac{(p+q)^2}{4} = 9$. Thus $a^2 = 5$, and $\{p,q\} = \{1,5\}$.

Choose $a = \sqrt{5}, k = 2\sqrt{5},$ and $p = 1$ for the endpoint B on minor arc AD. Then

$$\overrightarrow{OA} = (-\sqrt{5}, -2\sqrt{5}),$$
$$\overrightarrow{OB} = \left(-\frac{2}{\sqrt{5}}, -\frac{11}{\sqrt{5}}\right).$$

The sine of the central angle subtending minor arc AB is the absolute determinant of these radius vectors divided by 5^2, namely

$$\sin \widehat{AB} = \frac{|11-4|}{25} = \frac{7}{25}.$$

Hence $mn = 7 \cdot 25 = 175.$

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